Exercise 4

(a) Find the slope of the tangent line to the curve $y = x - x^3$ at the point $(1, 0)$

(i) using Definition 1 (ii) using Equation 2

(b) Find an equation of the tangent line in part (a).

(c) Graph the curve and the tangent line in successively smaller viewing rectangles centered at $(1, 0)$ until the curve and the line appear to coincide.

Solution

Definition 1 and Equation 2 give two ways of calculating the slope of a tangent line.

$$\text{Definition 1:} \quad m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\text{Equation 2:} \quad m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Part (a)

The function in this exercise is $f(x) = x - x^3$, and the value of x that we want to know the tangent line at is $a = 1$. Find m with Definition 1.

$$\begin{aligned} m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{(x - x^3) - (1 - 1^3)}{x - 1} &= \lim_{x \to 1} \frac{x - x^3}{x - 1} \\ &= -\lim_{x \to 1} \frac{x^3 - x}{x - 1} \\ &= -\lim_{x \to 1} \frac{x(x + 1)(x - 1)}{x - 1} \\ &= -\lim_{x \to 1} x(x + 1) \\ &= -1(1 + 1) \\ &= -2 \end{aligned}$$

$$\begin{aligned} m = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} &= \lim_{h \to 0} \frac{[(1 + h) - (1 + h)^3] - (1 - 1^3)}{h} \\ &= \lim_{h \to 0} \frac{(1 + h) - (1 + 3h + 3h^2 + h^3)}{h} \\ &= \lim_{h \to 0} \frac{-2h - 3h^2 - h^3}{h} \\ &= \lim_{h \to 0} (-2 - 3h - h^2) = -2 - 0 - 0 = -2 \end{aligned}$$

Part (b)

The aim is to find the equation of the line with slope 2 that passes through $(1, 0)$. Start with the general formula of a line.

$$y = mx + b$$

Since the slope is -2, $m = -2$.

$$y = -2x + b$$

Use the fact that the line passes through $(1, 0)$ to determine b.

$$0 = -2(1) + b$$

$$0 = -2 + b$$

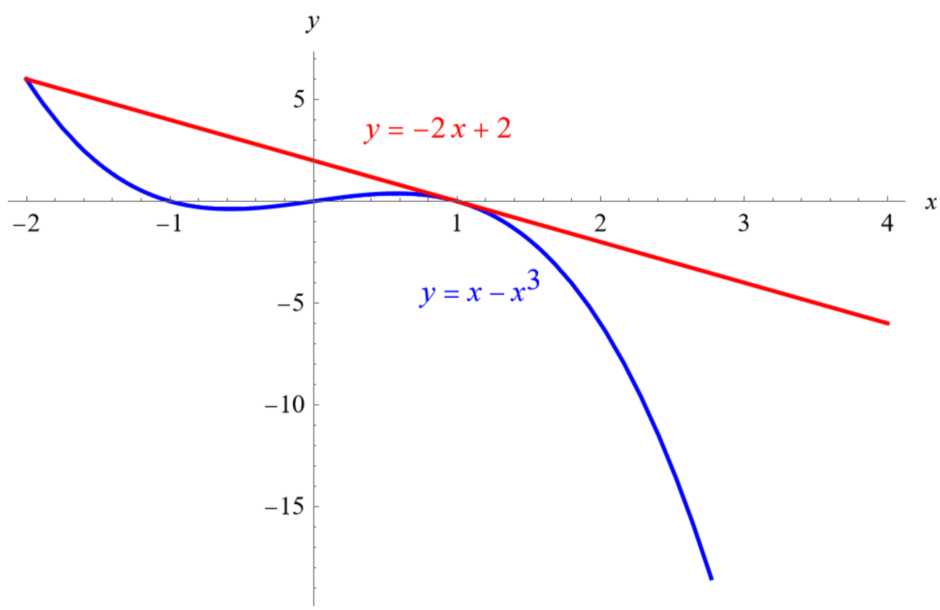
$$b = 2$$

Therefore,

$$y = -2x + 2.$$

Part (c)

Below is a plot of the parabola with the tangent line at $x = 1$.



Zoom in to the interval $0.9 \leq x \leq 1.1$.